ME 321: FLUID MECHANICS-I

Dr. A.B.M. Toufique Hasan
Professor
Department of Mechanical Engineering
Bangladesh University of Engineering and Technology (BUET), Dhaka

Lecture-07

31/01/2024

Fluid dynamics

- Linear Momentum Equation
- Bernoulli Equation

toufiquehasan.buet.ac.bd
toufiquehasan@me.buet.ac.bd

Recap

Reynolds transport theorem (RTT) for a fixed, nondeforming control volume (CV)

$$\boxed{\frac{d}{dt}\left(B_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\,\rho\,d\forall\right) + \int_{\text{CS}} \beta\,\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

This relation permits to change from a system approach to control volume approach.

where

$B_{\text{syst}} =$ any property of fluid (mass, momentum, enthalpy, etc.)

$\beta =$ intensive property of fluid (per unit mass basis)

$\rho =$ density of fluid

$d\forall =$ elemental volume

$\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA =$ elemental volume flux

$\int_{\text{CV}} =$ volume integral over the control volume (CV)

$\int_{\text{CS}} =$ surface integral over the control surface (CS)

Similar expression adopted by other books:

$$\boxed{\frac{D}{Dt}\left(B_{\text{syst}}\right) = \frac{\partial}{\partial t}\left(\int_{\text{CV}} \beta\,\rho\,d\forall\right) + \int_{\text{CS}} \beta\,\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

Conservation of linear momentum

$$B = m\vec{\mathbf{V}} \text{ (momentum)}, \qquad \beta = \frac{m\vec{\mathbf{V}}\text{ (momentum)}}{m \text{ (mass)}} = \vec{\mathbf{V}}$$

RTT takes the form of

$$\boxed{\frac{d\left(m\vec{\mathbf{V}}\right)_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \vec{\mathbf{V}}\rho \, d\forall + \int_{\text{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

$$\boxed{\vec{\mathbf{V}} = u\hat{i} + v\hat{j} + w\hat{k} = (u, v, w)}$$

Newton's second law of motion for a system is

Time rate of change of the linear momentum of the system = Sum of external forces acting on the system

$$\boxed{\frac{d\left(m\vec{\mathbf{V}}\right)_{\text{sys}}}{dt} = \sum \vec{F}_{\text{sys}} = \sum \vec{F}_{\substack{\text{contents of the}\\ \text{control volume}}}}$$

Since when a control volume coincident with a system at an instant of time, the forces acting on the system and the forces acting on the contents of the coincident control volume are instantaneously identical.

Conservation of linear momentum

For a control volume (CV) which is fixed and nondeforming, the Newton's second law of motion takes the following form:

$$\boxed{\sum \vec{F}_{\substack{\text{contents of the}\\ \text{control volume}}} = \frac{d}{dt}\int_{\mathrm{CV}} \vec{\mathbf{V}}\rho\, d\forall + \int_{\mathrm{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

Force contents of the control volume (CV) **=** Time rate of change of the linear momentum of the contents of the control volume (CV) **+** Net rate of linear momentum through the control surface (CS)

In fluid mechanics, there are two types of forces are to be considered,

(i) surface force, F_S which acts on the surfaces on the CV (pressure and shear stress)

(ii) body force, F_B which acts on the mass content of the CV (weight, electromagnetic force, etc.)

Then,

$$\boxed{\sum\left(\vec{F}_S + \vec{F}_B\right) = \frac{d}{dt}\int_{\mathrm{CV}} \vec{\mathbf{V}}\rho\, d\forall + \int_{\mathrm{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

**** Vector equation**

This is known as **momentum equation**
Or **equation of motion** in Fluid dynamics

Conservation of linear momentum

The momentum equation is a vector equation. Considering 3 components in Cartesian coordinate system (x, y, z), the momentum equation comes as:

$$x-\text{direction} \;:\; \sum\left(F_{S_x}+F_{B_x}\right)=\frac{d}{dt}\int_{\text{CV}} u\rho\, d\forall + \int_{\text{CS}} u\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$y-\text{direction} \;:\; \sum\left(F_{S_y}+F_{B_y}\right)=\frac{d}{dt}\int_{\text{CV}} v\rho\, d\forall + \int_{\text{CS}} v\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$z-\text{direction} \;:\; \sum\left(F_{S_z}+F_{B_z}\right)=\frac{d}{dt}\int_{\text{CV}} w\rho\, d\forall + \int_{\text{CS}} w\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

Applications of linear momentum equations will be discussed in later classes.

$$\boxed{\vec{\mathbf{V}}=u\hat{i}+v\hat{j}+w\hat{k}=(u,v,w)}$$

Differential Control Volume Analysis

Now, a **differential control volume** is considered. Using this approach, the differential equations can be obtained which will describe a flow field.

The differential control volume is fixed in space and bounded by flow streamlines, is thus an element of a stream tube as shown in Fig. The length of the control volume is ds.

Because the control volume is bounded by streamlines, the flow across the bounding surfaces occurs only at the end sections.

Apply the continuity and momentum equations considering:

(i) Steady flow
(ii) Inviscid flow (no friction, ideal fluid flow, μ = 0)
(iii) Incompressible flow (density is constant)
(iv) Irrotational flow (zero vorticity)
(v) Flow along a streamline

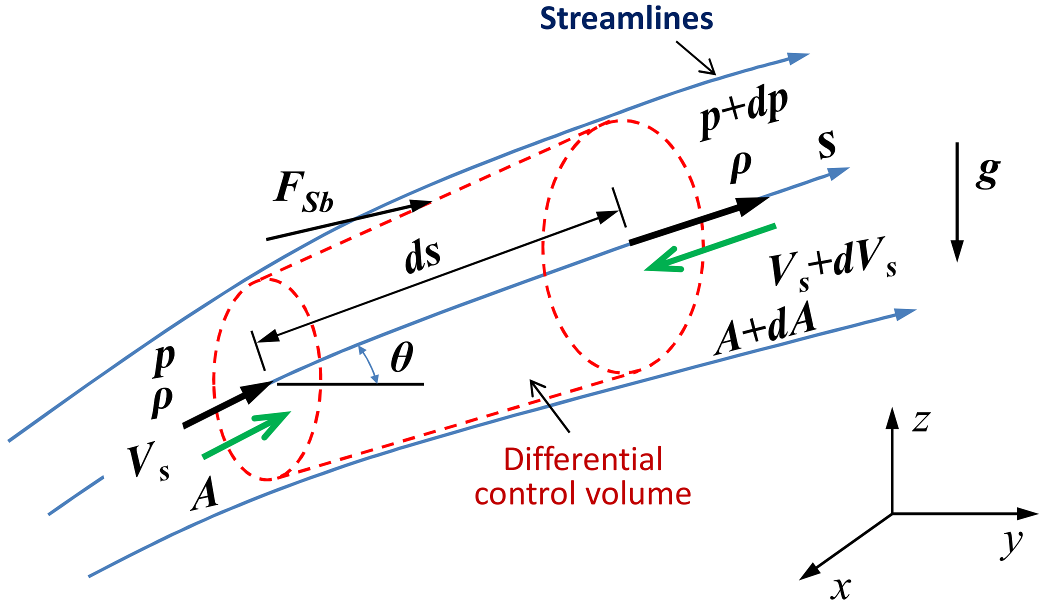

Fig. Differential control volume for momentum analysis of flow through a stream tube

Differential Control Volume Analysis

a. Continuity equation

$$\cancelto{=0}{\frac{d}{dt}\int_{CV} \rho \, d\cancel{V}} + \int_{CS} \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}\right) dA = 0$$

$$\Rightarrow \int_{CS} \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}\right) dA = 0$$

$$\Rightarrow -\rho V_s A + \{\rho (V_s + dV_s)(A + dA)\} = 0$$

$$\Rightarrow -\rho V_s A + \rho (V_s + dV_s)(A + dA) = 0$$

$$\Rightarrow -\rho V_s A + \rho V_s A + \rho V_s dA + \rho A dV_s + \cancelto{=0}{\rho dA dV_s} = 0$$

product of two differentials $dAdV_s$ is insignificant compared to other terms.

$$\boxed{\Rightarrow V_s dA + A dV_s = 0}$$

continuity equation for the differential control volume

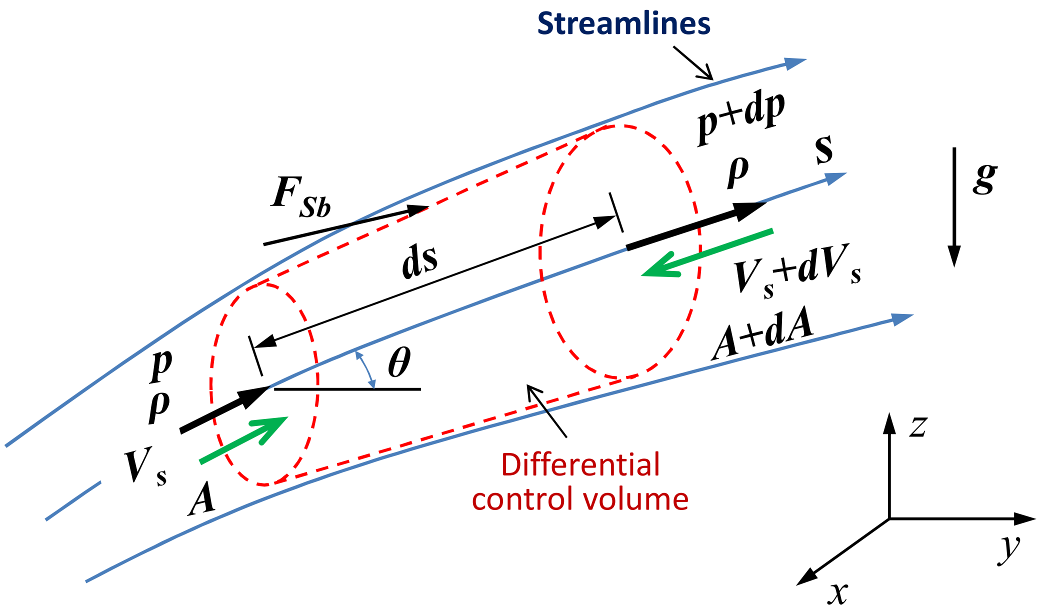

Fig. Differential control volume for momentum analysis of flow through a stream tube

b. Momentum equation (along steamwise direction)

$$\sum\left(\vec{F}_S+\vec{F}_B\right)=\cancelto{0}{\frac{d}{dt}\int_{\mathrm{CV}}\vec{\mathbf{V}}\rho\, d\mathcal{V}}+\int_{\mathrm{CS}}\vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow\ \sum\left(F_{S_s}+F_{B_s}\right)=\int_{\mathrm{CS}}V_s\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

(along streamwise direction, s)

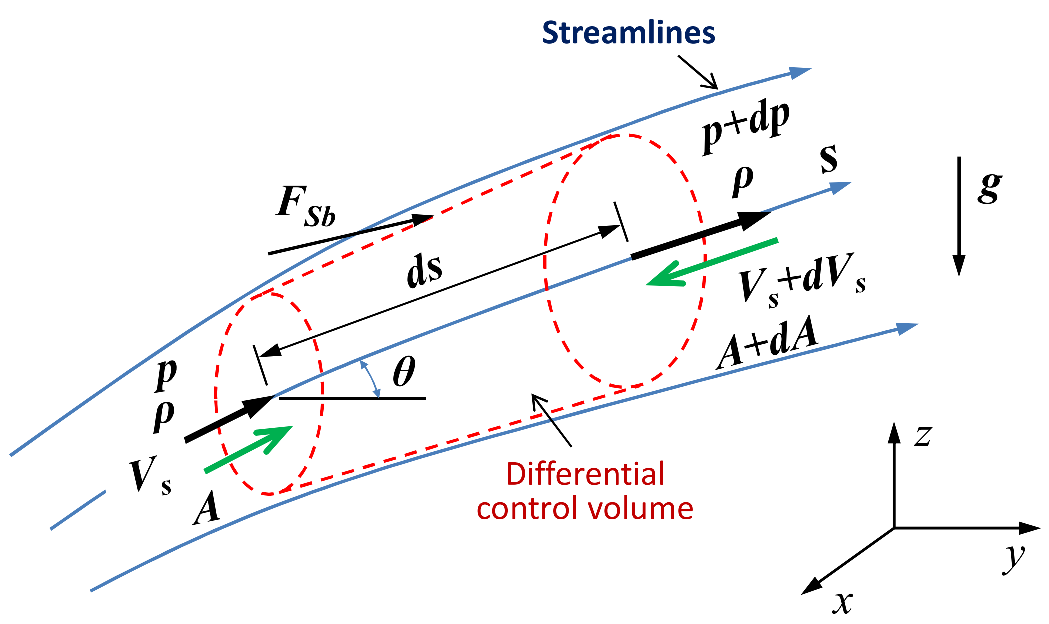

Fig. Differential control volume for momentum analysis of flow through a stream tube

Surface force only comes from the pressure (μ = 0):

$$F_{S_s}=pA-(p+dp)(A+dA)+\left(p+\frac{dp}{2}\right)dA$$

(Left face) (Right face) (bounding stream surface)

(average pressure acting on the bounding surface times the area component of the stream surface in s-direction, dA)

$$\Rightarrow\ F_{S_s}=pA-pA-pdA-Adp-\cancelto{0}{dpdA}+pdA+\cancelto{0}{\frac{dp}{2}dA}$$

product of two differentials $dpdA$ is insignificant compared to other terms.

$$\boxed{\Rightarrow\ F_{S_s}=-Adp}$$

Body force acting along *s*-direction:

$$F_{B_s} = \rho\left(-g\sin\theta\right)\left(A+\frac{dA}{2}\right)ds$$

$$\Rightarrow F_{B_s} = -\rho g\left(A+\frac{dA}{2}\right)dz$$

$$\Rightarrow F_{B_s} = -\rho g A dz - \rho g \underbrace{\cancel{\frac{dA}{2}dz}}_{=0}$$

$$\boxed{\Rightarrow F_{B_s} = -\rho g A dz}$$

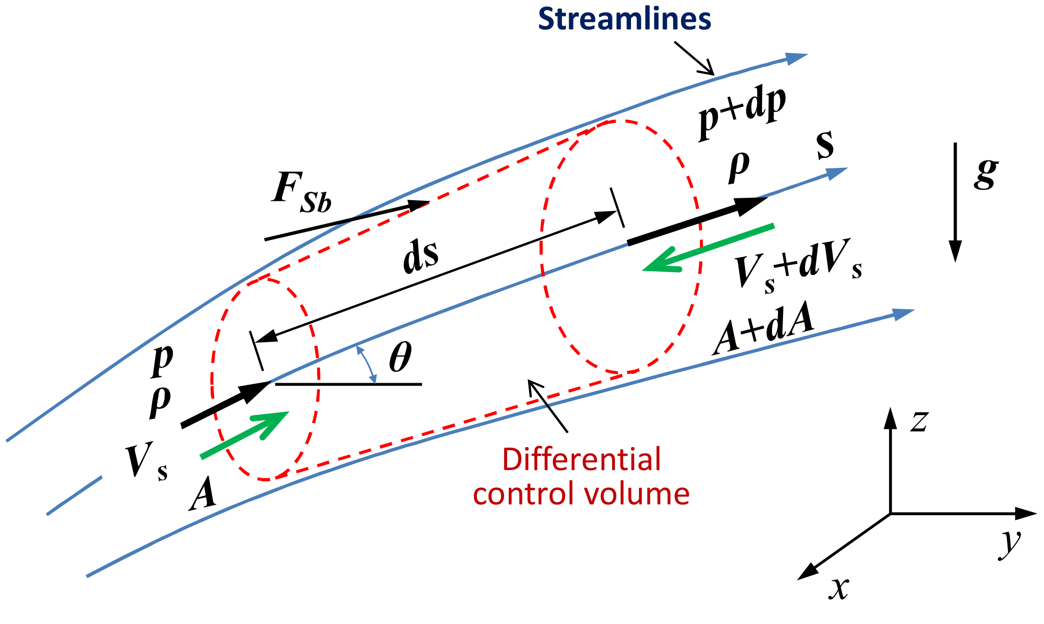

Fig. Differential control volume for momentum analysis of flow through a stream tube

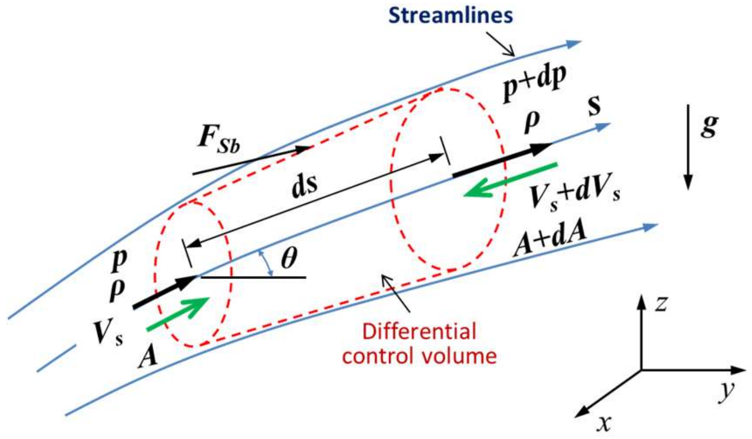

Fig. Differential control volume for momentum analysis of flow through a stream tube

Right hand side of momentum equation:

$$\int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = V_s\left(-\rho V_s A\right) + \left(V_s + dV_s\right)\left\{\rho\left(V_s + dV_s\right)\left(A + dA\right)\right\}$$

$$\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = -\rho V_s^{\,2} A + \left(V_s + dV_s\right)\left(\rho V_s A + \underbrace{\rho V_s dA + \rho A dV_s}_{=0} + \cancelto{=0}{\rho dV_s dA}\right)$$

$$\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = -\rho V_s^{\,2} A + \left(V_s + dV_s\right)\left(\rho V_s A\right)$$

$$\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = -\rho V_s^{\,2} A + \rho V_s^{\,2} A + \rho V_s A dV_s$$

$$\boxed{\Rightarrow \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = \rho V_s A dV_s}$$

$$V_s dA + A dV_s = 0 \ \text{(continuity)}$$
$$\therefore\ \rho V_s dA + \rho A dV_s = 0$$

Differential Control Volume Analysis

Now, momentum equation (along steamwise direction)

$$\sum\left(F_{S_s}+F_{B_s}\right)=\int_{CS} V_s\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow -Adp-\rho gAdz=\rho V_s AdV_s$$

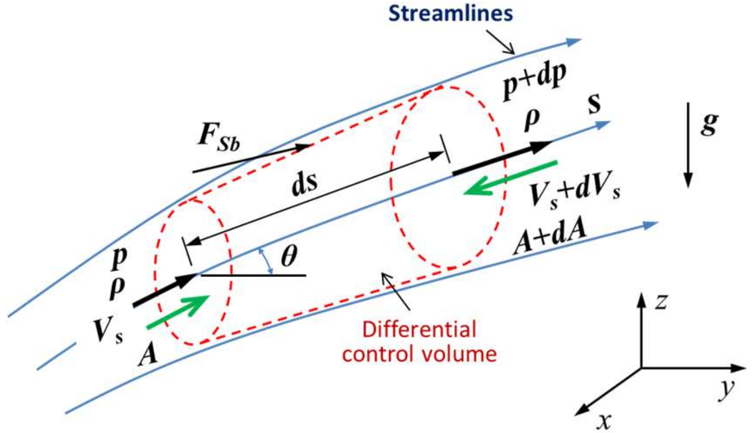

Fig. Differential control volume for momentum analysis of flow through a stream tube

Dividing both sides by ρA:

$$\Rightarrow -\frac{dp}{\rho}-gdz=V_s dV_s=d\left(\frac{V_s^{\,2}}{2}\right)$$

$$\boxed{\Rightarrow \frac{dp}{\rho}+d\left(\frac{V_s^{\,2}}{2}\right)+gdz=0}$$

Euler differential equation

for steady inviscid incompressible fluid flow .

Differential Control Volume Analysis

Integrate the Euler equation along a streamline:

$$\int \frac{dp}{\rho} + \int d\left(\frac{V_s^{\,2}}{2}\right) + \int g dz = \text{Constant}$$

$$\boxed{\Rightarrow \frac{p}{\rho} + \frac{V_s^{\,2}}{2} + gz = \text{Constant}}$$

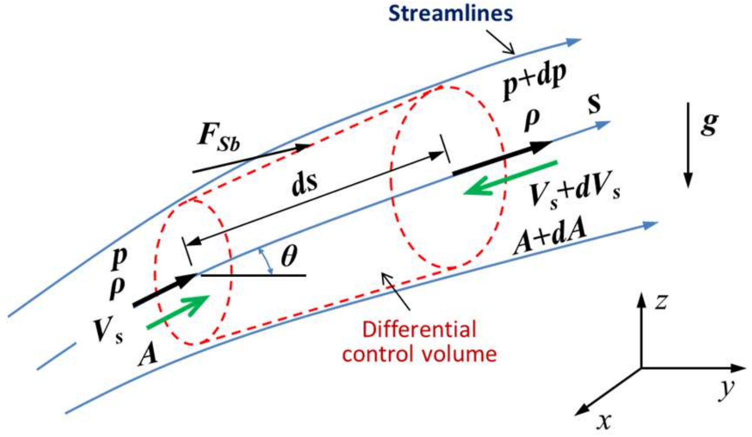

Fig. Differential control volume for momentum analysis of flow through a stream tube

Suffix *s* can be dropped conveniently (since fluid flows along the streamline)

$$\boxed{\Rightarrow \frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{Constant}}$$

Bernoulli equation.

(Most famous and mostly used equation in fluid dynamics)

Bernoulli equation

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{Constant}$$

Pressure head

Velocity head

Potential head

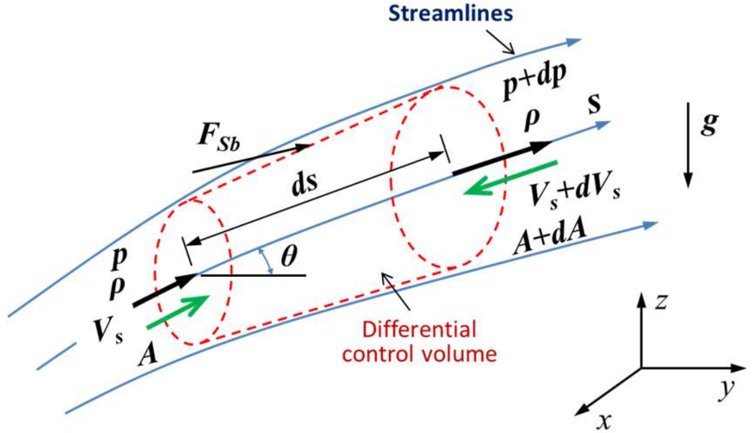

Fig. Differential control volume for momentum analysis of flow through a stream tube

Subjected to the following restrictions in fluid flow:

(i) Steady flow
(ii) Inviscid flow (no friction, ideal fluid flow, $\mu = 0$)
(iii) Incompressible flow (density is constant)
(iv) Irrotational flow
(v) Flow along a streamline

Considering any two points along a streamline, **Bernoulli Equation** yields:

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2}$$

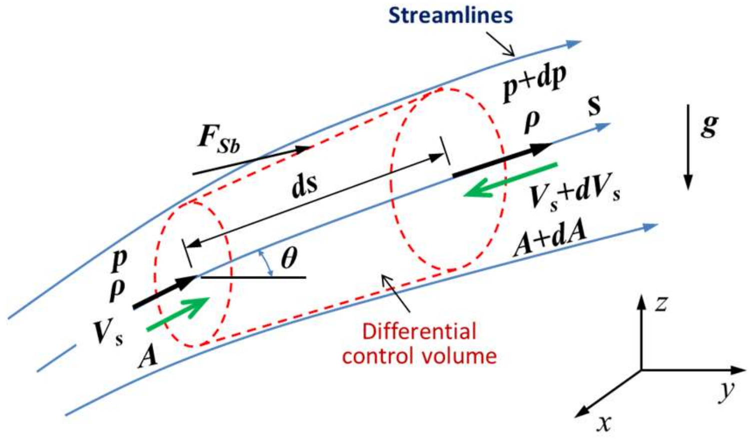

Fig. Differential control volume for momentum analysis of flow through a stream tube